

Complexity of Cylindrical Algebraic Decompositions via Regular Chains

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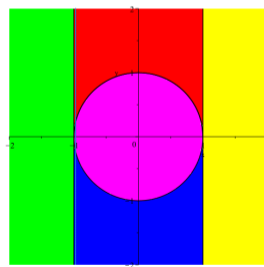
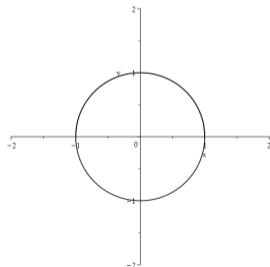
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What is a CAD?

A *cylindrical algebraic decomposition* (CAD) is a method in real algebraic geometry to decompose \mathbb{R}^n into a finite number of disjoint cells, each homeomorphic to some \mathbb{R}^k , over which a given set of polynomials have constant sign. Given such a decomposition it is easy to give a solution of a system of inequalities and equations defined by the polynomials.

It has applications in areas such as robotics and air traffic control problems, and was first discovered by George E. Collins in 1973, as part of his work on an effective method for quantifier elimination in real closed fields (a concept involving replacing a formula with an equivalent one without quantifiers).



Definitions

A cylindrical algebraic decomposition is in fact a decomposition that is both cylindrical and algebraic:

Definition

- **Decomposition:** Let $X \subseteq \mathbb{R}^n$. A *decomposition* of X is a finite collection of disjoint cells whose union is X .
- **Cylindrical:** A decomposition is *cylindrical* if for all $1 \leq j < n$, the projections on the first j variables of any two cells are either equal or disjoint.
- **Algebraic:** A decomposition is *algebraic* if each of its cells is a *semi-algebraic set*, i.e. a set that can be constructed by finitely many unions, intersections and complementations on sets of the form $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$, where $f \in \mathbb{R}[x_1, \dots, x_n]$.

Definition

Let $X \subseteq \mathbb{R}^n$ and $\mathbb{R}[x_1, \dots, x_n] \ni \mathcal{F} = \{f_i \in \mathbb{R}[x_1, \dots, x_n], 1 \leq i \leq r\}$, we say X is \mathcal{F} -invariant if each $f_i(x)$ has constant sign for every $x \in X$, that is, $\forall x \in X : f_i(x) \sigma_i 0$ for $\sigma_i = \{>, =, <\}$.

The CAD Algorithm

Collins' original CAD method is based on a projection and lifting scheme (PL-CAD):

Projection: Repeatedly apply a projection operator $Proj$:

$$F_n(x_1, \dots, x_n) \xrightarrow{Proj} F_{n-1}(x_1, \dots, x_{n-1}) \xrightarrow{Proj} \dots \xrightarrow{Proj} F_1(x_1)$$

Lifting:

- The real roots of the polynomials in F_1 plus the open intervals between them form an F_1 -invariant CAD of \mathbb{R}^1 .
- For each cell C of the F_{k-1} -invariant CAD of \mathbb{R}^{k-1} , isolating the real roots of the polynomials of F_k at a sample point of C , produces all the cells of the F_k -invariant CAD of \mathbb{R}^k above C .

A Different CAD Algorithm

There has been much work by Chen and Moreno Maza on an incremental algorithm for computing CADs using triangular systems and regular chains (RC-CAD), by creating a complex cylindrical tree and refining it into a real cylindrical tree.

A lot has been learnt about PL-CAD over the last forty years, with many enhancements made along the way. However, there has been a lot of work implementing RC-CAD into *Maple* via the `RegularChains` package, little analysis has been done. Therefore we are interested in trying to understand how the algorithm works in sufficient detail to give estimates and boundaries for its complexity, and in learning when and where the algorithm is more, or less, efficient.

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